Ring :-

Let R be a set and let two binary operations called addition (denoted by $+$) and multiplication (denoted by $\cdot$) be defined over the set R. Then the system $(R, +, \cdot)$ is called a ring R if the following conditions are satisfied

① Laws of addition:

(i) $a+b \in R$; $a, b \in R$ (Closure law)

(ii) $(a+b)+c = a+(b+c)$; $a, b, c \in R$ (Associative law)

(iii) There exists an element 0 in R called zero of the ring such that $a+0 = 0+a = a$, $\forall a \in R$

(iv) For each element a in R there exists $-a$ in R is called the negative of a such that

$$a+(-a) = (-a)+a = 0$$

(v) $a+b = b+a$; $a, b \in R$ (Commutative law)

② Laws of multiplication:

(i) $a \cdot b \in R$; $a, b \in R$ (Closure law)

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; $a, b, c \in R$ (Associative law)

③ Distributive law:

(i) $a \cdot (b+c) = a \cdot b + a \cdot c$

(ii) $(b+c) \cdot a = b \cdot a + c \cdot a$

(A) Commutative Ring :- A ring R is said to be commutative if $a \cdot b = b \cdot a$; $a, b \in R$

(B) Ring with unity :- If there exists an element 1 in R such that $1 \cdot a = a \cdot 1 = a$, $\forall a \in R$ then the ring R is called ring with unity element 1.

Elementary Properties of a ring :-

Theorem :- If R is a ring, then for all $a, b, c \in R$

(i) $a0 = 0a = 0$

(ii) $a(-b) = -(ab) = (-a)b$

(iii) $(-a)(-b) = ab$

(iv) $a(b-c) = ab - ac$

(v) $(b-c)a = ba - ca$

Proof :- (i) We have,

$$a0 = a(0+0) \qquad \{\because 0+0=0\}$$

$$= a0 + a0 \qquad \{\text{By left distributive law}\}$$

$$\therefore 0 + a0 = a0 + a0 \qquad \{\because a0 \in R \text{ and } 0 + a0 = a0\}$$

~~Now R is a group with respect to addition~~

$$\Rightarrow 0 = a0 \quad \text{—(i)} \quad \{R.C.L.\}$$

Similarly, we have

$$0a = (0+0)a$$

$$\Rightarrow 0a + 0 = 0a + 0a$$

$$\Rightarrow 0 = 0a \quad \text{—(ii)} \quad \{L.C.L.\}$$

From (i) & (ii), $a0 = 0a = 0$ Proved.

(ii) We have,

$$a\{(-b)+b\} = a0$$

$$\Rightarrow a(-b) + ab = 0 \qquad \{L.D.L.\}$$

$$\Rightarrow a(-b) = -(ab) \quad \text{—(i)}$$

Similarly,

we have,

$$(-a+a)b = 0b$$

$$\Rightarrow (-a)b + ab = 0$$

$$\Rightarrow (-a)b = -ab \quad \text{—(ii)}$$

From (i) and (ii) $\Rightarrow$ $a(-b) = -(ab) = (-a)b$.

(iii) we have,

$$(-a)(-b) = -[(-a)b] = -[-(ab)]$$

$$\Rightarrow (-a)(-b) = ab.$$

(iv) we have, $a(b-c) = a[b+(-c)]$

$= ab + a(-c)$ $\{L.D.L.\}$

$= ab + [-(ac)]$ $\{a(-c) = -(ac)\}$

$= ab - ac$

(v), we have, $(b-c)a = [b+(-c)]a$

$= ba + (-c)a$ $\{R.D.L.\}$

$= ba - ca$. $\{(-c)a = -(ca)\}$.

Proved

Example :- The set $R = \{0,1,2,3,4,5\}$ is a commutative ring with respect to '$+_6$' and '$\times_6$' as the two ring compositions.

Solution:-

The composition table for R for $+_6$

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4.

$\therefore$ R is an abelian group w.r.to $+_6$.